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TRANSITION SURFACES FOR SHORT-WAVE VIBRATIONS OF AN ELLIPSOIDAL SHELL IN A FLUID*

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The problem of resonance short-wave vibrations of a closed ellipsoidal elastic shell of revolution in a fluid is examined. An asymptotic solution (with respect to a large frequency parameter) is constructed for the Helmholtz equation for the acoustic pressure mode matched to the solutions of the system of equations of relative shell displacement. Is is shown that the radiation of a vibrating shell is governed by the location of the transition surfaces (TS) in the fluid and the transition lines on the shell that separate the non-wave zone with intensive pressure damping in the neighbourhood of the shell from the remote slowly damping radiation field. Regularities are investigated for the motion of the TS as the vibration frequency, circumferential wave number, and degree of curvature (prolateness) of the shell change. Estimates are given of the applicability of the asymptotic approach.

It is known that the vibrations of shells of revolution of non-constant curvature in a vacuum occur with the formation of transition lines in certain frequency bands, where the nature of the state of shell stress and strain changes /1/. It is natural to expact that this property is conserved even for shell vibrations in a fluid, where TS also originate in the fluid because of the connectedness between the shell and fluid vibration modes.

Interest in studying TS in fluids gave rise to a tendency to construct a solution of the problem that is valid both in the fluid layer surrounding the shell and in the far field. If radiation is not taken into account, then a problem is obtained that is similar to problems on shell vibrations in an incompressible fluid, whose solution for the pressure in the medium does not change its nature with distance from the shell. This problem is solved effectively by approximate analytical methods (see /2/, say). However, such a problem leaves open questions about the amplitudes of the resonant vibrations modes of the "shell-fluid" system, their damping because of energy radiation, and on the radiation field itself. Knowledge of the location of the TS as the connecting link between the near and far pressure fields in the fluid is the key to the solution of these questions within the framework of approximate methods.

1. We examine the short-wave quasitransverse vibrations of a closed prolate ellipsoidal shell of revolution submerged in an infinite compressible fluid excited by a normal periodic load $Q(\eta)\exp i(m\beta-\omega t)$. We separate the time component $\exp(-i\omega t)$ and we write the initial equations for the shell steady vibration modes in the fluid and of the fluid in spheroidal coordinates ξ,η,β $(1\leqslant\xi<\infty,\ |\eta|\leqslant 1,\ 0\leqslant\beta<2\pi)$

$$\begin{aligned}
&h_*^2\Delta_2^2w-\Delta_1X-\lambda^2w+P(\xi_0,\eta,\beta)=Q(\eta)e^{im\beta}\\
&\Delta_2^2X+\Delta_1 w=0,\ \Delta P+k^2P=0,\ w=2EhW\\
&H_3^{-1}(\xi_0,\eta)P_{,\xi}|_{\xi=\xi_0}=\lambda^2gw,\ \lambda^2=\omega^2\rho_0E^{-1},\ k=\omega c^{-1}\\
&H_3=\frac{d}{2}\left(\frac{\xi^2-\eta^2}{\xi^2-1}\right)^{1/2},\quad g=\frac{1}{2h}\frac{\rho}{\rho_0}\\
&h_*^2=\frac{h^2}{3(1-\nu^2)},\quad (\ )_{,\xi}=\frac{\partial}{\partial\xi}(\ )
\end{aligned}\tag{1.1}$$

Here W,h are the deflection and half-thickness of the shell whose surface in ξ,η,β coordinates agrees with the coordinate surface $\xi=\xi_0$, E,ρ_0,ν are Young's modulus, density, and Poisson's ratio of the shell material, ω,m are the angular frequency and wave number in the circumferential direction of the external vibrational load distributed along the shell meridian according to the law $Q(\eta)$ X is the stress function, P,ρ,c are the pressure, density, and velocity of sound in the fluid, Δ,Δ_2 are Laplace operators in space and on the shell

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surface, Δ_1 is the Vlasov operator, and d is the focal length of the ellipsoid $\xi = \xi_0$.

We note that the construction of an exact solution for the problem formulated in series in spheroidal wave functions is possible in practice only for small values of the frequency parameter $p = kd/2$; as p grows the series convergence is degraded sharply /3/.

An asymptotic method is proposed below for constructing the integrals of the initial equations for large p corresponding to the frequency range for short-wave shell vibrations in the fluid in which separation of all the functions in (1.1) into two groups is possible: slowly and rapidly varying along the meridian coordinate η. Among the slowly varying functions are the Lamé parameters and the radii of curvature of the shell, while the deflection, stress function, and pressure in the fluid are among the rapidly varying functions.

We will represent the integrals of the Helmholtz equation (1.1) in the form

$$P(\xi, \eta, \beta, p) = R(\xi, \eta, p)\, S(\eta, p) \exp(im\beta) \tag{1.2}$$

with the unknown functions $R(\xi, \eta, p)$ and $S(\eta, p)$ where $R(\xi, \eta, p)$ is considered to vary slowly along the coordinate η. The slow variability of the function R along the meridian coordinate is introduced in order to satisfy all the conditions (1.1) on the shell surface in the principal term of the asymptotic form P. This is possible for complete separation of variables only for shells in the shape of a cylinder or a sphere.

We set (1.2) in correspondence with (1.1) and we differentiate, taking the slow variability of part of the functions in the variable η into account. We arrive at a system of equations in $S(\eta, p)$ and $R(\xi, \eta, p)$

$$\begin{aligned} &(1-\eta^2) S_{,\eta\eta} + [\varkappa(\eta) - m^2(1-\eta^2)^{-1} - p^2\eta^2] S = 0 \\ &[(\xi^2 - 1) R_{,\xi}]_{,\xi} - [\varkappa(\eta) + m^2(\xi^2-1)^{-1} - p^2\xi^2] R = 0 \end{aligned} \tag{1.3}$$

in which the slowly varying function $\varkappa(\eta)$ is introduced in place of the separation constant. Solutions of this quasiseparated system should be valid in the whole space occupied by the fluid, including the shell surface on which they must be matched with the solutions of the surface (1.1).

We first consider matching the integrals of system (1.1), (1.3) oscillating in the meridian direction. As the forcing frequency approaches the resonance frequency of the shell-fluid system, these integrals determine mainly the vibration mode of a closed shell, while they actually agree at the resonance frequency. The influence of the force itself on the vibration mode and radiation field decreases here. The contribution from the vibration mode to the radiation can turn out to be predominant at the resonance frequency.

In the case of vibrations of a prolate ellipsoidal shell of revolution, the condition of oscillation along the shell meridian is solved firt in the less curved equatorial domain (near $\eta = 0$). We consider this to refer indeed to the appropriate integrals of the first equation of (1.3) on the shell surface. Hence, to construct oscillating integrals of system (1.3), the separation function $\varkappa(\eta)$ should be sought in the class of complex functions satisfying the conditions

$$\varkappa = \varkappa_1 + i\varkappa_2, \ \varkappa_1(0) - m^2 > 0, \ \ |\varkappa_2(0)|/\varkappa_1(0) \ll 1 \tag{1.4}$$

For $|\eta| > 0$ the nature of the integrals $S(\eta, p)$ remains primarily oscillating up to values $\eta = \pm\eta_*$ that make the real part of the coefficient of S in (1.3) zero

$$\varkappa_1(\eta_*) - m^2(1-\eta_*^2)^{-1} - p^2\eta_*^2 = 0 \tag{1.5}$$

For $|\eta| > \eta_*$ the integrals $S(\eta, p)$ become rapidly attenuating. The values $\eta = \pm\eta_*$ are turning points for the first equation in (1.3).

In parallel with the conditions (1.4) and (1.5), we also impose the requirement that the real part of the coefficient vanishes for the undifferentiated term in the second equation in (1.3) in a certain set of values of η from the interval $[-1, 1]$:

$$\varkappa_1(\eta) + m^2[\xi_*^2(\eta) - 1]^{-1} - p^2\xi_*^2(\eta) = 0 \tag{1.6}$$

This requirement yields a turning point in the second equation in (1.3) and sets up a connection between $\varkappa_1(\eta)$ and a certain surface $\xi = \xi_*(\eta) > \xi_0$, later called the transition surface (TS). The surfaces $\eta = \pm\eta_*$ will also be TS.

Taking (1.5) and (1.6) into account, we reduce (1.3) to the form of equations with one large parameter p

$$\begin{aligned} &(1-\eta^2) S_{,\eta\eta} + [p^2\varphi_1(\eta) + i\varkappa_2(\eta)] S = 0 \\ &[(\xi^2-1) R_{,\xi}]_{,\xi} - [p^2\varphi_2(\xi, \eta) + i\varkappa_2(\eta)] R = 0 \\ &\varphi_1 = (\xi_*^2(\eta) - \eta^2)[1 - \varphi_*/(1-\eta^2)(\xi_*^2(\eta) - 1)] \\ &\varphi_2 = (\xi_*^2(\eta) - \xi^2)[1 + \varphi_*/(\xi^2 - 1)(\xi_*^2(\eta) - 1)] \\ &\varphi_* = (1-\eta_*^2)(\xi_*^2(\eta_*) - 1), \ p^2\varphi_* = m^2 \end{aligned} \tag{1.7}$$

For any η, ξ from the domain of their definition the first equation in (1.7) can have not more than two turning points, and the second not more than one.

We will examine the solution of the second equation in (1.7) that yields the nature of the emitting component of the pressure in the fluid with distance from the shell. According to /4/, the uniform asymptotic form of the solution $R(\xi, \eta, p)$ is sought by using the linearly independent Airy functions Ai, Bi whose arguments and the weighting functions in front of them are determined from the system of recurrence equations for different powers of the large parameter p. The solution $R(\xi, \eta, p)$ should also satisfy the radiation condition at infinity and the condition of boundedness in strips of the spheroidal coordinate system. Omitting the intermediate computations, as carried out for example in /4/, we write down the principal term of the asymptotic form $R(\xi, \eta, p)$

$$
\begin{gathered}
R(\xi, \eta, p) = \Phi(\xi, \eta)\,[\mathrm{Bi}\,(p^{2/3}\Psi) - i\,\mathrm{Ai}\,(p^{2/3}\Psi)] + O(p^{-1}) \\
\Psi(\xi, \eta) = \operatorname{sign}(\xi_*(\eta) - \xi)\left(\frac{3}{2}\int_{\xi}^{\xi_*(\eta)} f_2^{1/2}(\xi, \eta)\,d\xi\right)^{2/3} \\
\Phi(\xi, \eta) = [(\xi^2 - 1)\,\Psi_{,\xi}(\xi, \eta)]^{-1/2}, \quad f_2 = \varphi_2(\xi, \eta)(\xi^2 - 1)^{-1}
\end{gathered}
\tag{1.8}
$$

(the branches are chosen such that $(-1)^{2/3} = 1$).

It is seen from (1.8) that during passage through the value $\xi = \xi_*(\eta)$ the arguments of the Airy function change sign from positive for $\xi < \xi_*(\eta)$ ($|\eta| < \eta_*$) to negative for $\xi > \xi_*(\eta)$, which results in a change in the asymptotic form of these functions: for a positive argument the function Bi grows as its modulus grows, while Ai decreases according to a law similar to an exponential low; both functions oscillate with slow damping for a negative argument.

Taking (1.8) into account, we set the solution (1.2) into the non-penetration condition of (1.1). We obtain a relationship between the pressure on the shell surface and the deflection

$$
\begin{gathered}
P(\xi_0, \eta, \beta) = -\lambda^2\mu(\eta)\,w(\eta, \beta), \\
\mu(\eta) = -gH_3(\xi_0, \eta)\,[R_{,\xi}(\xi, \eta)/R]_{\xi=\xi_0}^{-1}
\end{gathered}
\tag{1.9}
$$

If the surface $\xi = \xi_*(\eta)$ is not too close to the shell, (1.9) for $\mu(\eta)$ can be given the simpler form

$$
\mu(\eta) = \frac{gH_3(\xi_0, \eta)}{p f_2^{1/2}(\xi_0, \eta)}(1 + i\varepsilon(\eta)), \qquad \varepsilon = 4\exp\left(-2p\int_{\xi_0}^{\xi_*(\eta)} f_2^{1/2}(\xi, \eta)\,d\xi\right)
\tag{1.10}
$$

We call $\mu(\eta)$ the complex coefficient of the fluid apparent mass. Its real part in the asymptotic formula (1.10) is determined by the ratio between the exponentially damping component of the solution (1.8) and its derivative on the shell surface. The same result is obtained even when considering shell vibrations in a fluid without taking radiation into account, when the apparent mass coefficient corresponding to the oscillating integrals w is a real function of η. Therefore, the magnitude of the imaginary part of μ in (1.10) characterizes the emissive power of the shell vibration mode (within the limits of the applicability of this formula, i.e., for $\varepsilon \ll 1$), and depends on the location of the TS $\xi = \xi_*(\eta)$ and $\eta = \pm\eta_*$ in the fluid.

2. We determine the frequency band in which the representation is valid for the complex apparent mass of the fluid in the form (1.10), and for the regularities of the TS $\xi = \xi_*(\eta)$ and $\eta = \pm\eta_*$ "motions" in this band as the vibration frequency changes. To do this we satisfy the compatibility condition for the modes of the shell fluid short-wave vibrations.

We will write the Helmholtz equation in the short-wave approximation on the shell surface

$$
\Delta_2 P + H_3^{-2}P_{,\xi\xi} + k^2 P = 0, \quad \xi = \xi_0
$$

Replacing the second derivative with respect to ξ therein by non-differentiated terms and using (1.2) and (1.7), we obtain

$$
\begin{gathered}
\Delta_2 P + s(\eta)\,P = 0, \quad s(\eta) = a^2(\eta) + k^2 + i\delta(\eta) \\
a^2 = p^2 f_2(\xi_0, \eta)\,H_3^{-2}(\xi_0, \eta), \quad \delta = \varkappa_2(\eta)\,H_3^{-2}(\xi_0, \eta)\,(\xi_0^2 - 1)^{-1}
\end{gathered}
\tag{2.1}
$$

We note that (2.1) is equivalent to the first equation in (1.7) within the framework of the short-wave approximations.

Because of the proportionality between the integrals, $P(\xi_0, \eta, \beta)$ and $w(\eta, \beta)$ on the surface $\xi = \xi_0$ should simultaneously satisfy three equations in the two functions w and $\mathbf{X}$: these are the first two equations in (1.1) in which $P(\xi_0, \eta, \beta)$ is replaced by w, and (2.1). The condition for them to be compatible is the characteristic equation (R_1, R_2 are the principal radii of curvature of the shell)

$$
\begin{gathered}
\{h_*^2 s^2 - \lambda^2\,[1 + ga^{-1}(1 + i\varepsilon)]\}\,s^2 + (m^2 k_2 H_{20}^{-2} - sR_2^{-1})^2 = 0 \\
k_2 = R_2^{-1} - R_1^{-1}, \quad H_{20}^2 = (d/2)^2(1 - \eta^2)(\xi_0^2 - 1)
\end{gathered}
\tag{2.2}
$$

Expressing the characteristic index $s(\eta)$ in (2.2) in terms of the parameter $a(\eta)$, we obtain an algebraic equation of the 9-th degree in this parameter. Since the function $a(\eta)$ should be real, then we set up a connection between δ and ε

$$\delta = \lambda^2 g a^{-1} s_0^2 \varepsilon \varphi^{-1}(a), \quad s_0^2 = a^2 + k^2,$$
$$\varphi(a) = 2[2h_*^2 s_0^3 - \lambda^2 s_0(1 + ga^{-1}) - R_2^{-1}(m^2 k_2 H_{20}^{-2} - s_0 R_2^{-1})]$$

Then $a(\eta)$ is defined as the real positive root of (2.2) for $\varepsilon = \delta = 0$. The uniqueness of the positive root $a = a_1 > 0$ follows from the analysis carried out in /2/, where an analogous equation was obtained (with zero values of ε and δ) when taking into account only the rapidly damped components of the pressure function in the fluid at the shell surface.

Therefore, for small ε it is possible the TS coordinates in terms of the positive root a_1 of the algebraic equation of the 9-th degree obtained without taking radiation into account. Assuming that this root is known at each point of the shell meridian for a given frequency of vibration ω and a circumferential wave number m, we transform the third equation in (2.1) into a quadratic equation in $\xi_*^2(\eta)$:

$$x^2 - [1 + b_1(\eta) - c_1]x + b_1(\eta) - \xi_0^2 c_1 = 0, \quad x = \xi_*^2(\eta) \tag{2.3}$$
$$b_1(\eta) = \xi_0^2 + (a_1(\eta)/k)^2(\xi_0^2 - \eta^2), \quad c_1 = m^2 p^{-2}(\xi_0^2 - 1)^{-1}$$

Having determined $\xi_*^2(\eta)$, we find η_* as the positive root of the transcendental equation

$$(1 - \eta_*^2)[\xi_*^2(\eta_*) - 1] - (m/p)^2 = 0 \tag{2.4}$$

in the interval $[0, 1]$. Values of $\varepsilon(\eta)$ can be found afterward, and a decision on the applicability of (1.10) can be made according to their magnitude. If the quantity $\varepsilon(\eta)$ is of the order of one in a certain domain of variation of the coordinate η, then the more general representation (1.9) must be used to solve the initial problem in this domain.

The relationships obtained take the simplest form in the case of axisymmetric vibrations. The function $f_2(\xi, \eta)$ becomes equal to $|\xi_*^2(\eta) - \xi^2|/(\xi^2 - 1)$, while its integral in the exponent of the exponential (1.10) reduces to the difference of elliptic integrals of the first and second kinds. For $m = \varepsilon = 0$ the characteristic Eq.(2.2) is transformed into an equation of the fifth degree in a

$$[h_*^2(a^2 + k^2)^2 - (\lambda^2 - R_2^{-2})]a - \lambda^2 g = 0 \tag{2.5}$$

In the case of axisymmetric vibrations there are no TS $\eta = \pm\eta_*$; the expression

$$\xi_*(\eta) = [\xi_0^2 + (a_1(\eta)/k)^2(\xi_0^2 - \eta^2)]^{1/2} \tag{2.6}$$

is obtained for $\xi_*(\eta)$. It describes a symmetric ovaloid of revolution, tangent to the ellipsoid $\xi_* = \xi_0[1 + (a_1(0)/k)^2]^{1/2}$ in the equatorial plane, and to the ellipsoid $\xi_* = \xi_0[1 + (a_1(1)/k)^2(r_0/b_0)^2]^{1/2}$, at the vertices, where $r_0 = R_2(0)$ is the radius of the equator, and b_0 is the semimajor axis of the shell.

It is seen from (2.5) and (2.6) that for any fixed vibrations frequency, the positive root a_1 has a maximum at the shall equator (in conformity with the maximum curvature $R_2^{-1}(\eta)$) and decreases smoothly to the shell poles "adjusting" the TS $\xi = \xi_*(\eta)$ to them (in addition to the ratio $r_0/b_0 < 1$) Therefore, a general property of the TS in the fluid during axisymmetric vibrations is their substantially greater removal from the equator of the ellipsoidal shell then from its poles.

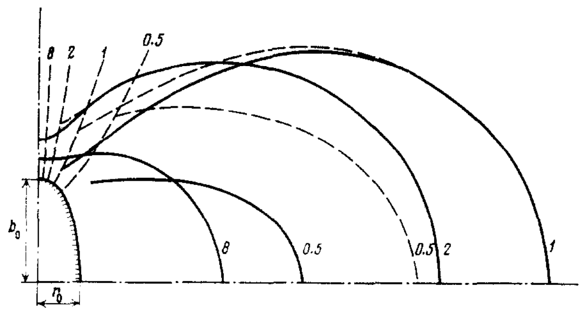
As the vibration frequency increases, the influence of the shell curvature on the root a_1 of (2.5) decreases. The additional adjustment of the TS to the shell poles is thereby reduced. The dissimilarity in the distances between the TS and the shell at the poles and at the equator is conserved exclusively because of the non-sphericity of the shell; the more prolate the shell, the closer does the TS approach its poles.

As the difference $\xi_*(\eta) - \xi_0$ decreases (for a fixed vibration frequency), the quantity $\varepsilon(\eta)$ increases. As follows from the asymptotic form of the solution (1.8), the pressure in the fluid, caused by the shell vibrations, damps out intensively in the domain between the shell and the TS. Behind the TS it is converted into a complex function that oscillates with slow damping, the radiation pressure. Hence, knowing the distance from points on the shell meridian to the TS in the fluid for the magnitude of the imaginary part of the complex apparent mass coefficient (ε) for a given vibration frequency, the degree of reduction of the pressure amplitude in the fluid can be determined up to the time it is transformed into radiation pressure.

It is clear from the above that the greatest radiation during axisymmetric vibrations starts from the region of shell surfaces adjacent to the poles. As the vibrations frequency decreases and (or) the prolateness of the shell increases, the contrast between radiation from the poles and from the equatorial region of the shell is magnified. Equilibrium of the radiation fields from individual sections of the shell surface occurs as the frequency increases.

The deductions made agree with the well-known representation about the connection between the elastic strain wavelength and the radiation in the region of the equator where the shell stiffness is minimal, the strain wavelength is also less than in the pole domain. The radiation from the shell equatorial domain should also be correspondingly less.

As an illustration, the TS coordinates are computed for axisymmetric vibrations of a steel ellipsoidal shell of revolution with $r_0/b_0 = 0.4$, $2h/r_0 = 0.01$ in water at frequencies related to the



curvature of the shell equator by $\lambda r_0 = n$, $n = 0.5$, $1, 2, 8$. The results are represented in the figure. The locations of the TS $\xi = \xi_*(\eta)$ for $n = 0.5, 1, 2, 8$ are shown by solid lines with appropriate notation. The outline of the shell meridian section is isolated by shading. Because of the symmetry of the pattern, only one quadrant is displayed. The quantities $\varepsilon = \varepsilon(n, \eta)$ were also computed to check the applicability of the asymptotic formula (1.10). In domains where the condition $\varepsilon \ll 1$ is violated, no TS were constructed (this is the domain near the shell poles for the curves with $n = 0.5$ and 1 in the figure). In these domains there are almost no zones of exponential damping. It can hence be assumed that the behaviour of the pressure characteristic for the far field is observed here directly from the shell surface, and the formula

$$\mu|_{\eta\approx 1} = p^{-2/3} g (d/2) (2\xi_0)^{-1/3} \operatorname{Ai}(0) [\operatorname{Ai}'(0)]^{-1} e^{i\pi/3}$$

can be used for the fluid apparent mass coefficient.

For non-axisymmetric vibrations the location of the TS $\xi = \xi_*(\eta)$ in the region of the equator is not different from the axisymmetric case for frequencies higher than or equal to the "annualar" frequency $(n \geqslant 1)$; at frequencies less than the annular frequency it is removed a greater distance from the equator. The dashed curves in the figure are for shell vibration modes with three waves in parallel. Near the shell vertices a TS $\eta = \pm\eta_*$ is formed for $m \neq 0$ (a two-sheeted hyperboloid of revolution), which intersects the shell along the transition lines. The location of the TS $\eta = \pm\eta_*$ for $m = 3$ and $n = 0.5, 1, 2, 8$ is also shown by dashes in the figure. The surface $\eta = \pm\eta_*$ isolates the shell vertex, creating a domain of acoustic shadow at the sheets of the hyperboloid of revolution. For a non-axisymmetric vibration mode the greatest radiation occurs from the shell surface sections adjacent to the hyperboloid $\eta = \pm\eta_*$ at the equator side, for which the distance between the shell and the TS $\xi = \xi_*(\eta)$ is minimal.

As follows from (2.4), as the variability of the vibration mode grows in the circumferential direction, the non-radiating domain expands on the shell surface. On the other hand, an increase in the vibrations frequency for a fixed number of waves along the parallels m displaces the TS $\eta = \pm\eta_*$ to the ellipsoid vertices.

Questions of constructing integrals of system (1.3) corresponding to complex values of $\varkappa$ different from (1.4) are not touched upon here. They are constructed by the above-mentioned method but considerably more simply: for them there is no need to determine the dependence of the separation of variables parameter $\varkappa$ on η. It is sufficient to assume that $\varkappa$ is constant. The integrals mentioned are utilized in constructing solutions of boundary value problems, and are, as a rule, localized at the shell boundaries, the lines of load application, and reinforcements.

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